KAZHDAN'S PROPERTY T AND C*-ALGEBRAS

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Abstract. Kazhdan's property T has recently been imported to the C*-world by Bekka. Our objective is to extend a well known fact to this realm; we show that a nuclear C*-algebra with property T is finite dimensional (for all intents and purposes). Though the result is not surprising, the proof is a bit more complicated than the group case.

1. Introduction

Kazhdan's revolutionary concept of property T has recently been translated into C*-language in [1]. One of the questions raised by Bekka's paper is whether or not one can generalize to the C*-context the classical fact that a discrete group which is both amenable and has property T must be finite (cf. [1, Proposition 11]). Unfortunately, the C*-situation is not quite as simple, but a satisfactory result can be obtained.

Theorem. *Let A be a unital C*-algebra which is both nuclear and has property T. Then $A = B \oplus C$ where B is finite dimensional and C has no tracial states.*

The irritating C-summand can't be avoided; if B is *any* C*-algebra with property T and C is *any* algebra without tracial states then $B \oplus C$ also has property T. Hence any finite dimensional C*-algebra plus a Cuntz algebra (for example) will have property T and be nuclear. On the other hand, the theorem above does imply that if A is nuclear, has property T and has a *faithful* trace then it must be finite dimensional – this is an honest generalization of the discrete group case since reduced group C*-algebras always have a faithful trace. More generally, every stably finite, nuclear algebra with property T is finite dimensional since Haagerup has shown that every unital stably finite exact C*-algebra must have a tracial state (cf. [5]). Since these are the main cases of interest, it seems fair to say *"property T plus amenability implies finite dimensional* (more or less)."

Perhaps the more interesting thing, however, is the proof. In the case of a discrete group it is trivial: amenability implies the left regular representation has almost invariant vectors; rigidity then provides a fixed vector; but, only finite groups have fixed vectors in the left regular representation.

Unfortunately, we have been unable to find a simple argument for the general case, hence the circuitous route taken here. Our approach requires generalizing Kazhdan projections, the theory of amenable traces and even the deep fact that nuclearity passes to quotients.

Finally, we express our gratitude to the reviewer for pointing out an error in our original manuscript; their careful reading greatly improved the truth of this work!

Partially supported by DMS-0244807.

2. Definitions and Notation

We make the blanket assumption that *all C*-algebras are unital and separable* unless otherwise noted or obviously false (e.g. $\mathbb{B}(\mathcal{H})$, the bounded operators on a (separable) Hilbert space $\mathcal{H}$, won't be norm separable).

Inspired by the von Neumann version (see, for example, [3]), Bekka defines property T for C*-algebras in terms of bimodules. If A is a C*-algebra and $\mathcal{H}$ is a Hilbert space equipped with commuting actions of A and its opposite algebra A^{op} then we say $\mathcal{H}$ is an A-A bimodule. (Another way of saying this is that there exists a $*$-representation $\pi\colon A \otimes_{\max} A^{op} \to \mathbb{B}(\mathcal{H})$.) As is standard, we denote the action by $\xi \mapsto a\xi b$, $\xi \in \mathcal{H}$, $a \in A$, $b \in A^{op}$. (That is, $a\xi b = \pi(a \otimes b)\xi$.)

Definition 2.1. A C*-algebra A has property T if every bimodule with almost central vectors has a central vector; i.e. if $\mathcal{H}$ is a bimodule and there exist unit vectors $\xi_n \in \mathcal{H}$ such that $\|a\xi_n - \xi_n a\| \to 0$ for all $a \in A$ then there exists a unit vector $\xi \in \mathcal{H}$ such that $a\xi = \xi a$ for all $a \in A$.

An important example of a bimodule is gotten by starting with an embedding $A \subset \mathbb{B}(\mathcal{K})$ and letting $\mathcal{H} = \mathrm{HS}(\mathcal{K})$ be the Hilbert-Schmidt operators on $\mathcal{K}$. The commuting actions of A and A^{op} are given by multiplication on the left and right: $T \mapsto aTb$ for all $T \in \mathrm{HS}(\mathcal{K})$, $a \in A$ and $b \in A^{op}$ (canonically identified, as normed involutive linear spaces, with A).

Throughout this note we will use Tr to denote the canonical (unbounded) trace on $\mathbb{B}(\mathcal{H})$ and, if $\mathcal{H}$ happens to be finite dimensional, tr will be the unique tracial state on $\mathbb{B}(\mathcal{H})$.

3. Kazhdan Projections

It is known that if Γ is a discrete group with property T then all the Kazhdan projections – the central covers in the double dual $C^*(\Gamma)^{**}$ coming from finite dimensional irreducible representations – actually live in $C^*(\Gamma)$. We extend this fact to the general C*-context.

Recall that an *intertwiner* of two $*$-representations $\pi\colon A \to \mathbb{B}(\mathcal{H})$ and $\sigma\colon A \to \mathbb{B}(\mathcal{K})$ is a bounded linear operator $T\colon \mathcal{H} \to \mathcal{K}$ such that $T\pi(a) = \sigma(a)T$ for all $a \in A$. We will need *Schur's Lemma.*

Lemma 3.1. *If two representations π and σ have a nonzero intertwiner and π is irreducible then π is unitarily equivalent to a subrepresentation of σ.*

The proof is simple, well-known and will be omitted – the main point is that irreducibility of π forces an intertwiner to be a scalar multiple of an isometry.

Property T groups are often defined as follows: if a unitary representation weakly contains the trivial representation then it must honestly contain it. Here is the generalization to our context.

Proposition 3.2. *Assume A has property T, $\pi\colon A \to \mathbb{M}_n(\mathbb{C})$ is an irreducible representation and $\sigma\colon A \to \mathbb{B}(\mathcal{K})$ is any representation which weakly contains π.*[1] *Then π is unitarily equivalent to a subrepresentation of σ.*

Proof. Let $\mathrm{HS}(\mathbb{C}^n, \mathcal{K})$ denote the Hilbert-Schmidt operators from $\mathbb{C}^n$ to $\mathcal{K}$. We make this space into an A-A bimodule by multiplication on the left and right – i.e. $aTb = \sigma(a)T\pi(b)$ for all $T \in \mathrm{HS}(\mathbb{C}^n, \mathcal{K}), a \in A, b \in A^{op}$. Since a nonzero central vector would evidently be an

[1]This means there exist isometries $V_k\colon \mathbb{C}^n \to \mathcal{K}$ such that $\|\sigma(a)V_k - V_k\pi(a)\| \to 0$ for all $a \in A$. This is equivalent to saying $\|V_k^*\sigma(a)V_k - \pi(a)\| \to 0$ for all $a \in A$, which explains the terminology 'weak containment'.

intertwiner of π and σ, it suffices (by property T and Schur's lemma) to show the existence of an asymptotically central sequence in $\mathrm{HS}(\mathbb{C}^n, \mathcal{K})$.

If $V_k\colon \mathbb{C}^n \to \mathcal{K}$ are isometries such that $\|\sigma(a)V_k - V_k\pi(a)\| \to 0$ then a routine calculation shows that the unit vectors $\frac{1}{\sqrt{n}}V_k \in \mathrm{HS}(\mathbb{C}^n, \mathcal{K})$ have the property that $\|a\frac{1}{\sqrt{n}}V_k - \frac{1}{\sqrt{n}}V_k a\|_{\mathrm{HS}} \to 0$ for all $a \in A$. $\square$

Recall that if $\pi\colon A \to \mathbb{B}(\mathcal{H})$ is a representation then there is a central projection $c(\pi) \in \mathcal{Z}(A^{**})$ in the double dual of A with the property that $c(\pi)A^{**} \cong \pi(A)''$ (among other things – see [8] for more). For property T groups and finite dimensional representations these projections are often called something else.

Definition 3.3. If A has property T and $\pi\colon A \to \mathbb{M}_n(\mathbb{C})$ is irreducible then the central cover $c(\pi)$ is also known as the *Kazhdan projection* associated to π.

One of the remarkable consequences of property T is that all Kazhdan projections actually live in (the center of) A (not just A^{**}). (Compare with the fact that $C^*(\mathbb{F}_2)$ has tons of finite dimensional representations, yet no nontrivial projections!)

Theorem 3.4. *Assume A has Kazhdan's property T. Then for each finite dimensional irreducible representation $\pi\colon A \to \mathbb{M}_n(\mathbb{C})$, the Kazhdan projection $c(\pi)$ actually lives in A.*

Proof. Let $\sigma\colon A \to \mathbb{B}(\mathcal{H})$ be a representation with the following three properties: (1) $\sigma(A)$ contains no nonzero compact operators, (2) $\pi \oplus \sigma\colon A \to \mathbb{B}(\mathbb{C}^n \oplus \mathcal{H})$ is faithful and (3) σ contains no subrepresentation which is unitarily equivalent to π. For example, one can start with a faithful representation of the algebra $(1 - c(\pi))A \subset A^{**}$ and inflate, if necessary, to arrange (1). (Standard theory of central covers shows that such a σ has no subrepresentation unitarily equivalent to π.)

Notice that such a representation σ can't possibly be faithful – if it were then Voiculescu's Theorem (cf. [4]) would imply that σ is approximately unitarily equivalent to $\sigma \oplus \pi$. In other words, σ weakly contains π and thus, by Proposition 3.2, actually contains π. This contradicts our assumption (3) and so σ can't be faithful.

Thus $J = ker(\sigma)$ is a nontrivial ideal in A. But assumption (2) implies that $\pi|_J$ must be faithful; hence, J is finite dimensional and has a unit p which is necessarily a central projection in A. We will show $p = c(\pi)$.

Since π is irreducible, $\pi(p) = 1$ and $J \cong \mathbb{M}_n(\mathbb{C})$. Hence we may identify the representations $A \to pA$ and π. Thus they have the same central covers – i.e. $p = c(\pi)$ as desired. $\square$

Here are a couple of consequences.

Corollary 3.5. *If A has property T then it has at most countably many non-equivalent finite dimensional representations.*

Proof. A separable C*-algebra has at most countably many orthogonal projections (since it can be represented on a separable Hilbert space). $\square$

Corollary 3.6. *Assume A has property T and let $J \triangleleft A$ be the ideal generated by all of the Kazhdan projections. Then A/J has no finite dimensional representations.*

Proof. Any nonzero, finite dimensional representation of A/J would produce a Kazhdan projection in A which wasn't in the kernel of the quotient map $A \to A/J$. $\square$

4. Amenable Traces

The notion of amenability for traces has a reasonably long history, with important contributions from several authors (see [2] for history and references). In this section we adapt one of Kirchberg's contributions (cf. [6]).

Definition 4.1. Let $A \subset \mathbb{B}(\mathcal{H})$ be a concretely represented unital C*-algebra. A tracial state τ on A is called *amenable* if there exists a state φ on $\mathbb{B}(\mathcal{H})$ such that (1) $\varphi|_A = \tau$ and (2) $\varphi(uTu^*) = \varphi(T)$ for every unitary $u \in A$ and $T \in \mathbb{B}(\mathcal{H})$.

It is a remarkable fact (due to Connes and Kirchberg) that this notion can be recast in terms of approximation by finite dimensional completely positive maps. See [2] or [6] for a proof of the theorem and [7] for more on completely positive maps.

Theorem 4.2. *Let τ be a tracial state on A. Then τ is amenable if and only if there exist unital completely positive maps $\varphi_n \colon A \to \mathbb{M}_{k(n)}(\mathbb{C})$ such that $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\|_{2,\mathrm{tr}} \to 0$, where $\|x\|_{2,\mathrm{tr}} = \sqrt{\mathrm{tr}(x^*x)}$, and $\tau(a) = \lim_{n\to\infty} \mathrm{tr} \circ \varphi_n(a)$, for all $a, b \in A$.*

With this approximation property in hand, the following proposition is straightforward.

Proposition 4.3. *Let A be a C*-algebra with property T. Then A has an amenable trace if and only if A has a nonzero finite dimensional quotient.*

Proof. Evidently a finite dimensional quotient yields an amenable trace (since every trace on a finite dimensional C*-algebra is amenable). Hence we assume A has property T and an amenable trace τ.

Let $\varphi_n \colon A \to \mathbb{M}_{k(n)}(\mathbb{C})$ be unital completely positive maps as in Theorem 4.2. Invoking Stinespring's Theorem, we can find representations $\rho_n \colon A \to \mathbb{B}(\mathcal{H}_n)$ and finite rank projections $P_n \in \mathbb{B}(\mathcal{H}_n)$ such that φ_n can be identified with $x \mapsto P_n\rho_n(x)P_n$. Let

$$\rho = \bigoplus_{n\in\mathbb{N}} \rho_n \colon A \to \mathbb{B}(\bigoplus_{n\in\mathbb{N}} \mathcal{H}_n)$$

and regard $\mathrm{HS}(\bigoplus_{n\in\mathbb{N}} \mathcal{H}_n)$ as an A-A bimodule via $T \mapsto \rho(a)T\rho(b)$.

The identity $P_n\rho_n(a) - \rho_n(a)P_n = P_n\rho_n(a)(1-P_n) - (1-P_n)\rho_n(a)P_n$ together with an unenlightening calculation shows that

$$\frac{\|P_n\rho_n(a) - \rho_n(a)P_n\|_2}{\|P_n\|_2} = \Big(\mathrm{tr}(\varphi_n(aa^*) - \varphi_n(a)\varphi_n(a^*)) + \mathrm{tr}(\varphi_n(a^*a) - \varphi_n(a^*)\varphi(a))\Big)^{\frac{1}{2}},$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm induced by $\langle S, T\rangle = \mathrm{Tr}(T^*S)$. By the Cauchy-Schwarz inequality we have

$$\mathrm{tr}(\varphi_n(aa^*) - \varphi_n(a)\varphi_n(a^*)) \leq \|\varphi_n(aa^*) - \varphi_n(a)\varphi_n(a^*)\|_{2,\mathrm{tr}}$$

and hence

$$\frac{\|P_n\rho_n(a) - \rho_n(a)P_n\|_2}{\|P_n\|_2} \to 0,$$

for every $a \in A$. That is, $\mathrm{HS}(\bigoplus_{n\in\mathbb{N}} \mathcal{H}_n)$ has a sequence of asymptotically central unit vectors (namely, $\frac{1}{\|P_n\|_2}P_n$). Hence property T gives us a nonzero central vector $T \in \mathrm{HS}(\bigoplus_{n\in\mathbb{N}} \mathcal{H}_n)$ – i.e. a nonzero compact operator in the commutant of $\rho(A)$. Since T's spectral projections must also live in $\rho(A)'$ we get a finite rank projection in the commutant. Thus $\rho(A)$, and hence A, has a finite dimensional quotient. □

Since property T evidently passes to quotients, the following corollary is a consequence of the previous result and Corollary 3.6.

Corollary 4.4. *Assume A has property T and let $J \triangleleft A$ be the ideal generated by all the Kazhdan projections. Then A/J has no amenable traces.*

5. Nuclearity and Property T

We now have the ingredients necessary for the mundane fact that nuclear C*-algebras with property T are finite dimensional plus something traceless.

Theorem 5.1. *Assume A is nuclear and has property T. Then $A = B \oplus C$ where B is finite dimensional and C admits no tracial states.*

Proof. Let J be the ideal generated by the Kazhdan projections in A. Note that A/J has property T and is nuclear. Since every trace on a nuclear C*-algebra is amenable (cf. [2]), Corollary 4.4 implies that A/J is traceless. Thus it suffices to show $B = J$ is finite dimensional (since it would then have to be a direct summand and $C = A/J$ is traceless).

So, how to see that J is finite dimensional? Well, if it weren't then we could find integers $k(n)$ such that

$$J = \bigoplus_{n=1}^{\infty} \mathbb{M}_{k(n)}(\mathbb{C}),$$

the c_0-direct sum (sequences tending to zero in norm) and thus the multiplier algebra of J is equal to the algebra of bounded sequences. Hence there is a unital $*$-homomorphism

$$A/J \to \frac{\prod_{n=1}^{\infty} \mathbb{M}_{k(n)}(\mathbb{C})}{\bigoplus_{n=1}^{\infty} \mathbb{M}_{k(n)}(\mathbb{C})}.$$

However, the Corona algebra on the right is easily seen to have lots of tracial states and so we deduce that A/J has a tracial state. But this is silly, as observed in the preceding paragraph. □

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